System Structure and Singular Control*

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ABSTRACT

The general continuous-time linear-quadratic control problem is considered. It is shown that recently developed linear system theoretic properties and algorithms play an important role in solving this singular control problem.

1. INTRODUCTION

We consider the general semidefinite linear-quadratic control problem for continuous-time systems. This problem was considered in depth for discrete-time systems in [11], and it was shown that there is a strong and important interplay between the structural properties of the underlying linear dynamical

*This research was supported in part by NSF Grant ECS7908673 and by the Joint Services Electronics Program through AFOSR/AFSC under Contract No. F44620-71-C-0067.

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system and the various existence and uniqueness questions which arise in the optimization problem.

While the discrete-time problem has been satisfactorily resolved, the continuous-time case has received far less attention and is not well understood for singular situations. The basic difference is that for singular problems in continuous time, (measurable) optimal solutions do not necessarily exist. While it has been long recognized [9] that by extending the class of allowable controls to include distributions it is possible in principle to take care of this difficulty, only special cases have been considered previously. Issues such as closed-loop stability under generalized control inputs have barely been addressed.

In this paper we present a fairly complete theory of singular control for continuous-time systems. As in the discrete-time case, linear system structure plays a fundamental role. In fact, structural properties which do not arise in the discrete-time situation are shown to be of great importance in the continuous-time case.

The basic setup of the singular control problem is given in Section 2, and the important system notion of *weak unobservability* is introduced. This property, together with the dual notion of *strong reachability*, is explored in detail in Section 3. The class of allowable distributional inputs for the control problem is also defined. A newly defined subspace of the state space (distributionally weakly unobservable states) is then introduced, which coincides in a special case with a space recently introduced by Willems related to "almost invariant subspaces." In Section 4 we discuss the "right structure algorithm." It is shown that the subspaces of interest can be completely characterized by this algorithm. Moreover, by its use, the singular problem can be reduced to a related nonsingular problem. In Section 5 this structural relationship is exploited to solve the open-loop version of the singular problem. The infinite-horizon problem and stability issues are considered in Section 5. It is shown that, as in the discrete-time case [11], the notion of strong detectability is important here, and that the regular part of the optimal-control law has a feedback implementation which stabilizes the closed-loop system for an initially stabilizable and strongly detectable system.

2. SINGULAR OPTIMAL CONTROL PROBLEMS

The general semidefinite linear-quadratic control problem on a finite interval is defined as follows: *Given the differential equation*

$$\dot{x} = Ax + Bu \tag{2.1}$$

with initial state $x(0)=x_0$, *determine* $u:[0,T]\to\mathbb{R}^m$ *such that*

$$(2.2)\quad J(x_0,u,T):=\int_0^T[x'(t)Qx(t)+2u'(t)Sx(t)+u'(t)Ru(t)]\,dt$$

is minimal.

Here $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $x_0\in\mathbb{R}^n$, $T>0$, and Q, S, R are matrices of suitable dimensions such that

$$(2.3)\qquad M:=\begin{bmatrix} Q & S' \\ S & R\end{bmatrix}$$

is symmetric and nonnegative semidefinite.

Special cases of this general problem are the *nonsingular problem* $R>0$ (Notice that $R\geqslant 0$ follows from $M\geqslant 0$) and the *standard problem* $R>0$ and $S=0$.

The standard problem is well established (see [5]). The nonsingular problem can be reduced to the standard problem by a suitable state-feedback transformation of the form $u=Fx+v$. To the authors' knowledge there does not exist a general treatment of the singular problem, i.e., the case where R is allowed to be singular.

A slightly modified formulation [11] brings in a systemic flavor, which enables one to intuitively understand and guess various properties of the system. Decompose M as

$$(2.4)\qquad \begin{bmatrix} Q & S' \\ S & R\end{bmatrix}=\begin{bmatrix} C' \\ D'\end{bmatrix}\begin{bmatrix} C & D\end{bmatrix}.$$

This is possible, since $M\geqslant 0$. As in the discrete-time case [11], it is quite useful to introduce the artificial output $y=Cx+Du$ and consider the system Σ:

$$(2.5)\qquad \begin{aligned}\dot{x}&=Ax+Bu,\\ y&=Cx+Du.\end{aligned}$$

Then the expression (2.2) can be written as

$$(2.6)\qquad J(x_0,u,T)=\int_0^T|y|^2\,dt,$$

where $|\cdot|$ denotes the Euclidean norm. Since $R = D'D$, the problem is nonsingular iff D has full column rank, i.e., is left invertible. The problem is standard if, in addition, $D'C = 0$.

REMARK 2.7. The claim made before, that the nonsingular problem can be reduced to the standard problem can easily be verified in this formulation. In fact, substitution of $u = Fx + v$ into (2.5) yields $\Sigma_F: \dot{x} = (A + BF)x + Bv$, $y = (C + DF)x + Du$. The condition for the resulting system to be standard is $D'(C + DF) = 0$, which can be satisfied by $F := -(D'D)^{-1}D'C$.

For the standard problem it is known that an optimal control always exists and is unique. The optimal control is given by a time-variable feedback

$$u = F(t)x, \tag{2.8}$$

and the minimal value of J equals

$$V(x_0, T) := \min_u J(x_0, u, T) = x_0' P(T) x_0, \tag{2.9}$$

where $P(t)$ is a nonnegative semidefinite matrix which is the solution of a suitable Riccati equation. These results carry over to the general nonsingular case.

To illustrate the usefulness of the introduction of the system Σ (2.5) into the problem we consider the question of finding the set of initial values x_0 for which $V(x_0, T)$ is zero—or equivalently, the null space, or kernel, of $P(T)$. Obviously, $x_0 \in \operatorname{Ker} P(T)$ iff there exists an input $u:[0, T] \to \mathbb{R}^m$ such that $y(t) = 0$, $0 \leqslant t \leqslant T$. Hence, as in the discrete-time case [11], we make the following definition:

DEFINITION 2.10. A state $x_0 \in \mathfrak{X} := \mathbb{R}^n$ is *weakly unobservable* on $[0, T]$ if there exists an input $u:[0, T] \to \mathbb{R}^m$ such that $y(t) = 0$ for $0 \leqslant t \leqslant T$.

The space of weakly unobservable states can be shown to be independent of T, and algebraic algorithms can be given for its computation [11]. If we denote this space by $\mathcal{V}$, then we see that for nonsingular problems $\ker P(T) = \mathcal{V}$ is independent of T.

Easy examples show that in the singular case optimal controls no longer necessarily exist.

EXAMPLE 2.11. Consider the system

$$\dot{x} = u, \qquad y = x, \qquad x(0) = 1.$$

Then $J=\int_0^T x^2(t)\,dt$ can be made arbitrarily small by a suitable choice of u [e.g., if $u(t)=-1/\varepsilon$ for $0\leqslant t\leqslant\varepsilon$ and $u(t)=0$ for $t>\varepsilon$, then $J\leqslant\varepsilon/3$]. Also, it is clear that for no piecewise continuous u (nor for any measurable control) J can be made zero.

The example suggests the use of impulses as admissible controls. For instance, if we were to take $u=-\delta(t)$ (the Dirac delta function), then $x(t)=0$ for $t>0$ and $J=0$. A rigorous setup for this approach can be obtained in the framework of the distributions, or generalized functions, as introduced by L. Schwartz.

Notice that we cannot admit all impulses. If, e.g. in Example 2.11, we take $\dot{\delta}$ (the derivative of the delta function), then $x(t)=1+\delta(t)$, and therefore $x^2(t)$ and hence J are not defined. In the general problem we will restrict our inputs to those for which the output y is a regular function. This will be described in detail in the next section. It will turn out that within the class of distributions for which the output is regular, an optimal input exists and is unique (see Section 5) provided that the system Σ is left invertible.

COMMENT 2.12. The singular optimal-control problem for continuous-time systems has been studied before, either directly (see [1], [2], [9]) or as a limiting case of a singular perturbation problem (see [6], [14], [15]). Typically, in this literature, it is assumed that $S=0$ and $R=0$. In addition, in the cases where an optimal control is actually computed or obtained via a limit process, additional regularity assumptions are made, e.g. $Q>0$, or more generally, $V_0=\cdots=V_{k-1}=0$, $V_k>0$, where $V_i:=B'A'^iQA^iB$.

In this paper, we only assume left invertibility of Σ, or equivalently, the uniqueness of optimal controls (see [2] for a discussion of problems where one does not have uniqueness).

The use of distributional control in the singular optimal control problem has been suggested repeatedly (see [1], [9], [15], and in particular [12]), but never pursued in any detail.

3. WEAKLY UNOBSERVABLE AND STRONGLY REACHABLE STATES

As suggested by the previous section, we are going to allow impulsive controls as inputs. The natural mathematically rigorous set up for impulsive controls is distribution theory. When allowing general distributions we run into a number of technicalities, which can be resolved but which obscure the inherent algebraic structure of the concepts and results to be discussed in this section. In order to avoid these technical details we restrict ourselves to a special class of distributions which is sufficiently nice to allow us to make the treatment completely algebraic and at the same time large enough to be representative for the system's behavior under general distributions as inputs. First we give a description of general distributions, then we define the subclass of distributions we will restrict ourselves to.

The set of distributions defined on $\mathbb{R}$ with support on $[0,\infty)$ is denoted $\mathcal{D}'_+$. This set is closed under convolution. For a detailed description of properties of distributions we refer to [10].

Particular examples of elements of $\mathcal{D}'_+$ are the δ-distribution and its derivatives. Linear combinations of these particular distributions will be called *impulsive distributions*. In order to simplify the notation we denote convolution by juxtaposition like ordinary multiplication; we denote the delta distribution by 1 and its derivative by p. A constant multiple of the delta distribution will, if no confusion can arise, simply be denoted by that constant:

$$\alpha\delta = \alpha\cdot 1 = \alpha.$$

An impulsive distribution can now be written as $\sum_{i=0}^{k} a_i p^i$, where $a_i \in \mathbb{R}$ for $i = 0,\ldots,k$, and where p^0 is understood to be the delta distribution 1.

Another particular class of elements of $\mathcal{D}'_+$ is the set of *regular* distributions in $\mathcal{D}'_+$. These are distributions that are functions. For most considerations the exact class of functions to be used to define regular distributions is not important; one can e.g. choose piecewise continuous, integrable, or measurable functions. In this paper, however, we make a much more restrictive assumption. We assume that our regular distributions $u(t)$ are *smooth on* $[0,\infty)$, i.e., that a function $v:[0,\infty)\to\mathbb{R}$ exists, arbitrarily often differentiable including at $t=0$, such that

$$u(t) = \begin{cases} 0 & (t<0), \\ v(t) & (t\geqslant 0). \end{cases}$$

Differentiability at $t=0$ of $v(t)$ is defined in an obvious way. Equivalently one can say that $v(t)$ can be extended to an arbitrarily often differentiable function defined on some interval $(-\varepsilon,\infty)$, where $\varepsilon > 0$.

Now we are in the position to define the class of distributions to be used in this section.

DEFINITION 3.1. An *impulsive-smooth* distribution is a distribution u of the form

$$u = u_1 + u_2$$

where u_1 is impulsive [i.e., $u_1 = \psi(p)$ for some polynomial ψ] and u_2 is smooth on $[0, \infty)$. The class of these distributions is denoted $\mathcal{C}_{\text{imp}}$.

Thus, regular distributions as well as impulsive distributions are in $\mathcal{C}_{\text{imp}}$. The following property of $\mathcal{C}_{\text{imp}}$ is crucial.

PROPOSITION 3.2. *$\mathcal{C}_{\text{imp}}$ is closed under convolution, in particular under differentiation (= convolution with p) and integration (= convolution with p^{-1}).*

Various properties of the class $\mathcal{C}_{\text{imp}}$ are discussed in [4]. In particular, it is shown in [4] that $u \in \mathcal{C}_{\text{imp}}$ is invertible (with respect to convolution) iff u is not in $C^{\infty}(\mathbb{R})$, i.e., u is not infinitely often differentiable in $t = 0$. For example, $u = p - a$ is invertible and $(p - a)^{-1} = e^{at}$ $(t \geqslant 0)$. More generally, if A is a $n \times n$ matrix, then $pI - A$ is invertible and $(pI - A)^{-1} = e^{tA}$ $(t \geqslant 0)$. Here and elsewhere in this paper we use the straightforward extension of distributional concepts to vectors and matrices.

Let us now consider the system Σ:

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{3.3}$$

in the above framework. We have to define what we mean by the solution of (3.3) with initial value x_0. This is a nontrivial matter, since distributions do not have a well-defined value at a particular time instant t_0. If we are interested in the particular case $x_0 = 0$, the situation is simple: we require that $x \in \mathcal{C}^n_{\text{imp}}$. For such x we have $\dot{x} = px$, and hence (3.3) yields

$$x = (pI - A)^{-1}Bu,$$

$$y = T(p)u,$$

where $u \in \mathcal{C}^m_{\text{imp}}$ and

$$T(s) := C(sI - A)^{-1}B + D. \tag{3.4}$$

[*Note*: $T(s)$ is a rational matrix in the indeterminate s; $T(p)$ is the matrix-valued distribution obtained by substituting $s = p$ and interpreting $(pI - A)^{-1}$ to be the convolution inverse of $pI - A$, whence $(pI - A)^{-1} = e^{tA}$ for $t \geqslant 0$).

It is well known that the solution x of (3.3) within the class $\mathcal{D}'_+$ is unique, and it follows from the foregoing considerations that $x \in \mathcal{C}^n_{\text{imp}}$ if $u \in \mathcal{C}^m_{\text{imp}}$.

If we want to define solutions with initial value x_0, we again require x to be in $\mathcal{C}^n_{\text{imp}}$, but we replace the equation $px = Ax + Bu$ by

$$px = Ax + Bu + x_0, \tag{3.5}$$

where x_0 stands for the $\mathbb{R}^n$-valued distribution $x_0 \cdot 1$. The addition of the term x_0 in the right-hand side results in a jump of the state variable x. In addition to this jump there may be other jumps caused by impulsive terms in u. However, if u is regular, then (3.5) implies $x(0+) := \lim_{t \downarrow 0} x(t) = x_0$, which is in accord with our intuitive idea of initial value. In the general case where u is allowed to have an impulsive component, $x(0+)$ consists of two terms, the "initial value" x_0 and a term resulting from the impulsive part of u.

When comparing this distributional setup for linear systems with the more conventional interpretation, we observe that a number of new concepts arise, not present for systems with only regular inputs. In particular, we may introduce the space of *instantaneously reachable* points. A point x_0 is called instantaneously reachable if there exists an input $u \in \mathcal{C}^m_{\text{imp}}$ such that with zero initial state, we have $x_0 = x(0+)$. It is easily seen that $x(0+)$ only depends on the impulsive part of u. If u is impulsive, say

$$u = \sum_{i=-k}^{0} u_i p^{-i}$$

then the equation $(pI - A)x = Bu$ implies that the impulsive part x_{imp} of x is given by

$$x_{\text{imp}} = \sum_{i=-k+1}^{0} x_i p^{-i},$$

where the coefficients x_i are determined by

$$x_{i+1} = Ax_i + Bu_i \qquad (i = -k, \ldots, -1) \tag{3.6}$$

and $x(0+)$ is the coefficient of p^{-1} in the expansion of $(pI - A)^{-1}Bu$ in powers of p^{-1}, i.e.

$$x(0+) = x_1 = Ax_0 + Bu_0.$$

We conclude that the coefficients of x satisfy the discrete-time equation corresponding to (3.3). Similarly we have

$$y_i = Cx_i + Du_i \qquad (i = k,\dots,0), \tag{3.7}$$

where the y_i's are coefficients in y_{imp}, the impulsive part of y.

As a consequence of this, we see that the space of instantaneously reachable states is the image of $[B, AB,\dots,A^{n-1}B]$, i.e., the (ordinary) reachable set.

In the optimization problem discussed in Section 2 we want to minimize $\int |y|^2\,dt$. Therefore we insist that the inputs be such that the output y is regular, i.e., $y_{\text{imp}} = 0$. Such inputs will be called *admissible*, and the space of admissible inputs, which is of course system dependent, is denoted $\mathcal{U}_\Sigma$. An explicit description of $\mathcal{U}_\Sigma$ will be given in Section 4.

The rest of this section will be devoted to a discussion of two important spaces $\mathcal{V}$ and $\mathcal{W}$.

DEFINITION 3.8. A state x_0 is called *weakly unobservable* if there exists a regular input u on $[0,\infty)$ such that the output resulting from x_0 and u is identically equal to zero on $[0,\infty)$. The space of weakly unobservable states is denoted $\mathcal{V} = \mathcal{V}(\Sigma)$.

The space $\mathcal{V}$ is easily seen to be a linear subspace of the state space $\mathcal{X} = \mathbb{R}^n$ containing the unobservable states.

We have the following simple property.

PROPOSITION 3.9. *Let $x_0 \in \mathcal{V}$, and let $u:[0,\infty) \to \mathbb{R}^m$ be a smooth input such that $y(t) = 0$ for $t > 0$. Then $x(t) \in \mathcal{V}$ for all $t > 0$.*

In fact, for any $t_1 > 0$ we can use the input $u_1(t) := u(t + t_1)$ $(t > 0)$ corresponding to the initial state $x_1 := x(t_1)$.

THEOREM 3.10. *$\mathcal{V}$ is the largest subspace $\mathcal{L}$ of $\mathcal{X}$ for which there exists a feedback $F: \mathcal{X} \to \mathbb{R}^m$ such that*

$$(A + BF)\mathcal{L} \subseteq \mathcal{L}, \qquad (C + DF)\mathcal{L} = 0. \tag{3.11}$$

Proof. We first show the following statement:

$$\forall x_0 \in \mathcal{V} \quad \exists u_0 \in \mathbb{R}^m \quad \left[Ax_0 + Bu_0 \in \mathcal{V}, \quad Cx_0 + Du_0 = 0\right]. \tag{3.12}$$

Let $x_0 \in \mathcal{V}$, and let u be such that $y(t) = 0$ for $t > 0$. According to Proposition 3.9 we have that $x(t) \in \mathcal{V}$ for all $t > 0$. Hence $\dot{x}(0) = Ax_0 + Bu_0 \in \mathcal{V}$, where $u_0 := u(0)$. In addition, $y(0) = Ax_0 + Bu_0 = 0$, which proves (3.12).

Now let $x_1, \dots, x_k$ be a basis of $\mathcal{V}$, and construct $u_1, \dots, u_k$ according to (3.12). Choose a map $F: \mathcal{X} \to \mathbb{R}^m$ such that $Fx_i = u_i$ $(i = 1, \dots, k)$. Then (3.12) translates to the following. For $i = 1, \dots, k$ we have

$$(A + BF)x_i \in \mathcal{V}, \qquad (C + DF)x_i = 0.$$

Since $x_1, \dots, x_k$ is a basis of $\mathcal{V}$, it follows that $\mathcal{L} = \mathcal{V}$ satisfies (3.11).

Now let $\mathcal{L}$ be any space for which there exists F such that (3.11) is satisfied. Then, if $x_0 \in \mathcal{L}$, there exists an input u of (3.3), viz. the feedback $u = Fx$, such that $y(t) = 0$ for all $t > 0$. Hence $x_0 \in \mathcal{V}$. It follows that $\mathcal{L} \subseteq \mathcal{V}$. ■

DEFINITION 3.13. A state x_1 is called *strongly reachable* (from the origin) if there exists an impulsive input $u \in \mathcal{U}_\Sigma$ such that for the corresponding state trajectory x we have $x(0+) = x_1$. The space of strongly reachable states is denoted $\mathcal{W} = \mathcal{W}(\Sigma)$.

Obviously, $\mathcal{W}$ is a linear subspace of $\mathcal{X}$ contained in the reachable space. If an impulsive $u \in \mathcal{U}_\Sigma$ gives $x(0+) = x_1$ for zero initial state, then the same control will produce the state $x(0+) = x_0 + x_1$ if the initial state is x_0. Therefore, the set of states instantaneously reachable from x_0 by means of admissible inputs is $x_0 + \mathcal{W}$. In particular we see that $\mathcal{W}$ is also the space of *strongly controllable states*, i.e. states from which the origin can be reached instantaneously by means of an admissible input.

The strongly reachable space can be given an interpretation in terms of the recurrence relations (3.6) and (3.7). The condition that y be regular translates to $y_i = 0$ $(i \leqslant 0)$. Therefore we find: A state $\tilde{x}$ is strongly reachable iff there exists an input sequence $(u_{-k}, \dots, u_0)$ such that with initial condition $x_{-k} = 0$ (3.6) and (3.7) yield sequences (x_i) and (y_i) satisfying

$$y_{-k} = \cdots = y_0 = 0 \quad \text{and} \quad x_1 = \tilde{x}.$$

It follows from these considerations that $\mathcal{W}$ satisfies the following condition:

$$\text{(3.14)} \quad \forall x_0 \in \mathcal{W} \quad \forall u_0 \in \mathbb{R}^m \quad \left[Cx_0 + Du_0 = 0 \;\Rightarrow\; Ax_0 + Bu_0 \in \mathcal{W} \right].$$

We use this property to prove the following result.

THEOREM 3.15. *There exists an "output injection" $G:\mathbb{R}^r \to \mathcal{X}$ such that*

$$(A+GC)\mathcal{W} \subseteq \mathcal{W}, \qquad \operatorname{im}(B+GD) \subseteq \mathcal{W}. \tag{3.16}$$

Moreover, $\mathcal{W}$ is the smallest subspace of $\mathcal{X}$ for which such a G can be found.

Proof. Choose a basis

$$\begin{bmatrix} x_1 \\ u_1 \end{bmatrix}, \dots, \begin{bmatrix} x_l \\ u_l \end{bmatrix} \tag{3.17}$$

of $\mathcal{W} \oplus \mathbb{R}^m$ such that the first k elements form a basis of $(\mathcal{W} \oplus \mathbb{R}^m) \cap \ker[C, D]$. Define

$$y_i := Cx_i + Du_i \qquad (i = 1, \dots, l).$$

Then $y_i = 0$ $(i = 1, \dots, k)$ and $y_{k+1}, \dots, y_l$ are independent. Choose $G:\mathbb{R}^r \to \mathcal{X}$ such that $Gy_i = -Ax_i - Bu_i$ for $i = k+1, \dots, l$. Then

$$w_i := [A + GC, B + GD] \begin{bmatrix} x_i \\ u_i \end{bmatrix} = Ax_i + Bu_i + Gy_i.$$

For $i \leqslant k$ we have $w_i = Ax_i + Bu_i \in \mathcal{W}$ by (3.14), and for $i > k$, $w_i = 0 \in \mathcal{W}$. Since (3.17) is a basis of $\mathcal{W} \oplus \mathbb{R}^m$, the result is proved.

Now let $\mathcal{L}$ be any subspace for which there exists a map $G:\mathbb{R}^r \to \mathcal{X}$ such that

$$(A+GC)\mathcal{L} \subseteq \mathcal{L}, \qquad \operatorname{im}(B+GD) \subseteq \mathcal{L}. \tag{3.18}$$

We show that we must have $\mathcal{W} \subseteq \mathcal{L}$. Let $\tilde{x} \in \mathcal{W}$. Then there exists a sequence $u_{-k}, \dots, u_0$ such that if $x_{-k} = 0$ and

$$x_{i+1} := Ax_i + Bu_i \qquad (i = -k, \dots, 0)$$

we have $y_i := Cx_i + Du_i = 0$ $(i = -k, \dots, 0)$ and $x_i = \tilde{x}$. It follows that

$$x_{i+1} = (A + GC)x_i + (B + GD)u_i.$$

But then (3.18) implies that $x_i \in \mathcal{L}$, $i = -k, \dots, 1$, in particular, $\tilde{x} = x_1 \in \mathcal{L}$. Hence $\mathcal{W} \subseteq \mathcal{L}$. ■

It follows from Theorems 3.10 and 3.15 that $\mathcal{V}$ and $\mathcal{W}$ are *dual* concepts. Specifically, if we define $\Sigma' := (A', C', B', D')$ (where the prime attached to a matrix denotes transposition), then

$$\mathcal{V}_{\Sigma'} = (\mathcal{W}_\Sigma)^\perp, \qquad \mathcal{W}_{\Sigma'} = (\mathcal{V}_\Sigma)^\perp.$$

It is possible to give algorithms which produce the spaces $\mathcal{V}$ and $\mathcal{W}$. An algorithm for $\mathcal{V}$ can be obtained from Theorem 3.10 and the condition (3.12). The latter condition can be written as

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V} \oplus 0_y) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix}, \tag{3.19}$$

where 0_y stands for the zero vector in y-space ($= \mathbb{R}^r$) and it follows from Theorem 3.10 that $\mathcal{V}$ is the largest space satisfying an inclusion like (3.19). If we define the sequence of spaces $\mathcal{V}_0, \dots, \mathcal{V}_n$ by

$$\mathcal{V}_0 := \mathcal{X}, \qquad \mathcal{V}_{i+1} := \begin{bmatrix} A \\ C \end{bmatrix}^{-1} \left\{ (\mathcal{V}_i \oplus 0_y) + \operatorname{im} \begin{bmatrix} B \\ D \end{bmatrix} \right\}, \tag{3.20}$$

then it is easily seen that $\mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \cdots$ and that $\mathcal{V}_i = \mathcal{V}_{i+1}$ implies $\mathcal{V}_i = \mathcal{V}_k$ $(k \geqslant i)$. Consequently $\mathcal{V}_{n-1} = \mathcal{V}_n$ and hence $\mathcal{V}_n$ satisfies (3.19). Also, if $\mathcal{L}$ is any space satisfying (3.19), then by induction we have $\mathcal{V}_i \supseteq \mathcal{L}$ for $i = 0, \dots, n$. Hence $\mathcal{V}_n = \mathcal{V}$. The spaces $\mathcal{V}_i$ can be given a system-theoretic interpretation: *$\mathcal{V}_i$ is the space of initial states x_0 for which there exists a regular input u such that for the resulting output y we have $y^{(j)}(0+) = 0$ $(j = 0, \dots, i-1)$.* It follows from the above considerations that, if we can find an input u such that $y^{(j)}(0+) = 0$ $(j = 0, \dots, n-1)$, then we also can find a regular input u such that $y(t) = 0$ for $t \geqslant 0$.

Dual results are valid for $\mathcal{W}$. In the first place (3.14) can be rewritten as

$$[A, B]\{(\mathcal{W} \oplus \mathbb{R}^m)\} \cap \ker[C, D] \subseteq \mathcal{W}, \tag{3.21}$$

and $\mathcal{W}$ is the smallest subspace satisfying such an inclusion. This suggests the recursion

$$\mathcal{W}_0 := 0, \qquad \mathcal{W}_{i+1} := [A, B]\{(\mathcal{W}_i \oplus \mathbb{R}^m) \cap \ker[C, D]\} \subseteq \mathcal{W}. \tag{3.22}$$

Here, $\mathcal{W}_i$ can be interpreted as the space of states x_1 strongly reachable by an impulsive input u of order $\leqslant i-1$ [i.e., u is of the form $u = \psi(p)$ where ψ is a polynomial of degree $\leqslant i-1$]. This interpretation follows from the considerations preceding (3.14). We have $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \cdots \subseteq \mathcal{W}_n$, and if $\mathcal{W}_i = \mathcal{W}_{i+1}$ then $\mathcal{W}_i = \mathcal{W}$—in particular, $\mathcal{W}_n = \mathcal{W}$.

Next we consider the spaces $\mathcal{V} + \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}$ and their relation to system invertibility.

PROPOSITION 3.23. *$x_0 \in \mathcal{V} + \mathcal{W}$ iff there exists $u \in \mathcal{U}_\Sigma$ such that $y(t) = 0$ $(t > 0)$.*

Proof. Let $x_0 = x_1 + x_2$ with $x_1 \in \mathcal{V}$, $x_2 \in \mathcal{W}$. There exists an impulsive u_2 such that $x(0+, x_2, u_2) = 0$. [By $x(t, x_0, u)$ we denote the value of the solution of (3.5) at $t > 0$ resulting from initial value x_0 and control u. Recall that $x(\cdot, x_0, u)$ is in $\mathcal{C}^n_{\text{imp}}$, so that $x(t, x_0, u)$ is defined for $t > 0$ and also $x(0+, x_0, u)$.] In addition, there exists a regular u_1 such that $y(t, x_1, u_1) = 0$ for $t \geqslant 0$. Using the input $u := u_1 + u_2 \in \mathcal{U}_\Sigma$ we have $y(t, x_0, u) = 0$.

Conversely, let for some $u \in \mathcal{U}_\Sigma$ and initial state x_0 the output $y(t)$ be identically zero. The input can be decomposed as $u = u_1 + u_2$, where u_1 is regular and u_2 is impulsive. Let $x(0+, x_0, u_2) =: x_1$. Then $x(t, x_1, u_2) = x(t, x_0, u)$, since $u = u_2$ for $t > 0$ and the state variables have the same value at $t = 0+$. Hence, $y(t, x_1, u_2) = y(t, x_0, u) = 0$, so that $x_1 \in \mathcal{V}$. Since $x_2 := x_0 - x_1$ in $\mathcal{W}$, we have $x_0 \in \mathcal{V} + \mathcal{W}$. ■

Because of Proposition 3.23 we call elements of $\mathcal{V} + \mathcal{W}$ *distributionally weakly unobservable*. The space $\mathcal{V} + \mathcal{W}$ is closely related to the right invertibility of Σ. We say that Σ is *right invertible* if for every regular function $y: [0, \infty) \to \mathbb{R}^r$ there exists an input $u \in \mathcal{U}_\Sigma$ such that, with the initial state $x_0 = 0$, the corresponding output trajectory equals y.

THEOREM 3.24. *The following statements are equivalent:*

(i) *Σ is right invertible.*

(ii) *$\mathcal{V} + \mathcal{W} = \mathcal{X}$ and* $\operatorname{im}[C, D] = \mathbb{R}^r$.

(iii) *The transfer function $T(s)$ (see* (3.4)*) is right invertible as a rational matrix.*

Proof. (i) ⇒ (ii): Since $y(t) \in \operatorname{im}[C, D]$, the condition $\operatorname{im}[C, D] = \mathbb{R}^r$ is obvious. Now we take any $x_0 \in \mathcal{X}$ and we show that $x_0 \in \mathcal{V} + \mathcal{W}$. Consider $y(t, x_0, 0) = Ce^{tA}x_0$. There exists $u \in \mathcal{U}_\Sigma$ such that $y(t, 0, u) = y(t, x_0, 0)$ for $t > 0$. Then we have that $y(t, x_0, -u) = y(t, x_0, 0) - y(t, 0, +u) = 0$ for $t > 0$. Since $-u \in \mathcal{U}_\Sigma$, we conclude from Proposition 3.23 that $x_0 \in \mathcal{V} + \mathcal{W}$.

(ii) ⇒ (iii): If $T(s)$ is not right invertible, there exists a nonzero polynomial row vector $\psi(s)$ such that $\psi(s)T(s) = 0$. Let $x_0 \in \mathcal{X} = \mathcal{V} + \mathcal{W}$. Then there exists $u \in \mathcal{U}_\Sigma$ such that $y(t, x_0, u) = 0$ $(t > 0)$, i.e.,

$$T(p)u + C(pI - A)^{-1}x_0 = 0.$$

Hence $\psi(p)C(pI - A)^{-1}x_0 = 0$ for all x_0, which implies that $\psi(p)C(pI - A)^{-1} = 0$ and consequently $\psi(p)C = 0$. By (3.4) it follows that $0 = \psi(p)T(p) = \psi(p)D = 0$. This is clearly in contradiction with $\mathrm{im}[C, D] = \mathbb{R}^r$.

(iii) ⇒ (i): If $T(s)$ is right invertible, there exists $R(s)$ such that $T(s)R(s) = I$. For any regular function y we choose $u = R(p)y$. Then

$$y(\cdot, 0, u) = T(p)u = y. \qquad \blacksquare$$

PROPOSITION (3.25). *$x_0 \in \mathcal{V} \cap \mathcal{W}$ iff there exists $u \in \mathcal{U}_\Sigma$ such that $x(0+, 0, u) = x_0$ and $y(t, 0, u) = 0$ for $t > 0$.*

Proof. If $x_0 \in \mathcal{V} \cap \mathcal{W}$, there exists an impulsive $u_1 \in \mathcal{U}_\Sigma$ such that $x_0 = x(0+, 0, u_1)$ and there exists a regular u_2 such that $y(t, x_0, u_2) = 0$ for $t \geqslant 0$. It follows that $u = u_1 + u_2$ satisfies the condition. The converse is straightforward.

A system is called *left invertible* if there exists no nonzero input $u \in \mathcal{U}_\Sigma$ such that the output $y(t, 0, u) = 0$ for $t \geqslant 0$. Since $y(\cdot, 0, p^{-k}u) = p^{-k}y(\cdot, 0, u)$ and since $p^{-k}u$ is regular for sufficiently high k, we may replace "$u \in \mathcal{U}_\Sigma$" by "regular u" in the definition of left singularity.

THEOREM 3.26. *The following statements are equivalent*:

(i) *Σ is left invertible.*

(ii) *$\mathcal{V} \cap \mathcal{W} = 0$ and* $\ker\begin{bmatrix} B \\ D \end{bmatrix} = 0$.

(iii) *The transfer function $T(s)$ is left invertible as a rational matrix.*

Proof. (i) ⇒ (ii): If $Bu_0 = 0$, $Du_0 = 0$ for some $u_0 \neq 0$, then, choosing the impulsive input $u = u_0$ $(= u_0 \cdot \delta)$, we find that $y(t, 0, u_0) = 0$ for all $t \geqslant 0$. Hence Σ is not invertible. Now suppose that $x_0 \in \mathcal{V} \cap \mathcal{W}$ and $x_0 \neq 0$. Then, according to Proposition 3.25, there exists $u \in \mathcal{U}_\Sigma$ such that $y(t, 0, u) = 0$ for all $t \geqslant 0$ and $x(0+, 0, u) = x_0$. The latter equality implies that $u \neq 0$, so that Σ is not left invertible.

(ii) ⇒ (iii): If $T(s)$ is not left invertible, there exists a nonzero vector-valued polynomial $\phi(s)$ such that $T(s)\phi(s) = 0$. Let $u := \phi(p)$ and $x_0 := x(0+, 0, u)$. Then $x_0 \in \mathcal{V} \cap \mathcal{W}$ because of Proposition 3.25, since $y(\cdot, 0, u) =$

$T(p)u = T(p)\phi(p) = 0$. Then by assumption $x_0 = 0$ and consequently $x(t) = 0$ for $t > 0$, since $u(t) = 0$ for $t > 0$. It follows that $(pI - A)^{-1}Bu = 0$ and hence $Bu = 0$. In addition, $T(p)u = 0$ and hence $Du = 0$ [see (3.4)]. Since $\ker B \cap \ker D = 0$, this implies that $u = 0$, which is a contradiction.

(iii) $\Rightarrow$ (i): If $R(s)$ is a left inverse of $T(s)$, then $T(p)u = 0$ implies $u = R(p)T(p)u = 0$. ∎

COMMENT 3.27. The spaces $\mathcal{V}, \mathcal{W}, \mathcal{V} \cap \mathcal{W}, \mathcal{V} + \mathcal{W}$ have been discussed by various authors, under varying conditions. The space $\mathcal{V}$ was given in [11] for discrete-time systems. In a paper also considering exclusively discrete-time systems, Molinari introduced the spaces $\mathcal{V}$ and $\mathcal{W}$ (see [7]). In particular the algorithms (3.20) and (3.22) are introduced in [7]. In [8] and [13, Problem 5.17] the spaces $\mathcal{V}$ and $\mathcal{W}$ are discussed for the special case that $D = 0$. While the generalization to the case where D is allowed to be nonzero is not very difficult (see, e.g. [13, Example 4.6], where a method is proposed to reduce, somewhat artificially, the general case to the case $D = 0$), it is very essential in such matters as invertibility and the singular optimal-control problem (compare [11]). Also, in [8] and [13], no (open-loop) systemic interpretations are given for these spaces. Such interpretations are given (for $D = 0$) in [12], in terms of almost invariant subspaces. Also, the use of distributions in order to describe various spaces was suggested in [12]. Note that $\mathcal{V}$, $\mathcal{V} + \mathcal{W}$, $\mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}$ coincide with the spaces $\mathcal{V}^*$, $\mathcal{V}_b^*$, $\mathcal{R}_b^*$, and $\mathcal{R}$, respectively, in the notation of [12].

4. APPLICATION OF THE STRUCTURE ALGORITHM

The space $\mathcal{V}$ can be computed by the structure algorithm as shown in [11]. For the computation of the space $\mathcal{W}$ we need a dual version of the structure algorithm. We discuss this dual algorithm in detail, for not only does it enable us to compute $\mathcal{W}$, but it will also be used for the transformation of the singular optimal-control problem into a nonsingular problem.

Consider the system

$$px = Ax + Bu + x_0, \qquad y = Cx + Du. \tag{4.1}$$

If D is not left invertible, there exists a basis transformation S in u-space such that D has the form $[\bar{D}, 0]$. Specifically

$$DS = [\bar{D}, 0], \tag{4.2}$$

where $\bar{D}$ is left invertible. Let

$$BS =: [\bar{B}, \tilde{B}], \tag{4.3}$$

and introduce a new control variable by

$$u = S\begin{bmatrix} \bar{u} \\ \tilde{u} \end{bmatrix}. \tag{4.4}$$

Then, (4.1) can be rewritten as

$$\begin{aligned} px &= Ax + \bar{B}\bar{u} + \tilde{B}\tilde{u} + x_0, \\ y &= Cx + \bar{D}\bar{u}. \end{aligned} \tag{4.5}$$

It follows from (4.5) that the output will be regular if $\bar{u}$ is regular and $\tilde{u}$ is the derivative of a regular function. This suggests the substitution

$$\tilde{u} = pv, \tag{4.6}$$

which yields

$$p(x - \tilde{B}v) = Ax + \bar{B}\bar{u} + x_0. \tag{4.7}$$

If we next substitute

$$x_1 := x - \tilde{B}v, \tag{4.8}$$

we obtain

$$\begin{aligned} px_1 &= Ax_1 + \bar{B}\bar{u} + A\tilde{B}v + x_0, \\ y &= Cx_1 + \bar{D}\bar{u} + C\tilde{B}v. \end{aligned} \tag{4.9}$$

Thus we have a new system Σ_1:

$$\begin{aligned} px_1 &= Ax_1 + B_1u_1 + x_0, \\ y &= Cx_1 + D_1u_1, \end{aligned} \tag{4.10}$$

where

$$B_1 := [\bar{B}, A\tilde{B}], \qquad D_1 := [\bar{D}, C\tilde{B}], \quad \text{and} \quad u_1' := [\bar{u}', v']. \tag{4.11}$$

Note that

$$u = P_0(p)u_1, \tag{4.12}$$

where

$$P_0(s) := S\begin{bmatrix} I & 0 \\ 0 & sI \end{bmatrix}. \tag{4.13}$$

Consequently, the transfer function of Σ_1 is

$$T_1(s) = T(s)P_0(s). \tag{4.14}$$

It follows that

$$u_1 \in \mathcal{U}_{\Sigma_1} \quad \Leftrightarrow \quad P_0(p)u_1 \in \mathcal{U}_{\Sigma}. \tag{4.15}$$

We claim that

$$\mathcal{W}_1 = \operatorname{im} \tilde{B} \tag{4.16}$$

[where $\mathcal{W}_1$ is defined as in Section 3; see (3.22)]. As a matter of fact, according to (3.22), $\mathcal{W}_1 = \{Bu | Du = 0\}$ and according to the above construction, this is im $\tilde{B}$.

We have the following relations between the weakly unobservable and the strongly reachable spaces of Σ and Σ_1:

PROPOSITION 4.17.

(i) $\mathcal{V}(\Sigma) \subseteq \mathcal{V}(\Sigma_1)$,
(ii) $\mathcal{W}(\Sigma) \supseteq \mathcal{W}(\Sigma_1)$,
(iii) $\mathcal{V}(\Sigma) + \mathcal{W}(\Sigma) = \mathcal{V}(\Sigma_1) + \mathcal{W}(\Sigma_1)$.

Proof. (i): $x_0 \in \mathcal{V}(\Sigma)$ iff there exists a regular u such that

$$T(p)u + C(pI - A)^{-1}x_0 = 0, \tag{4.18}$$

and $x_0 \in \mathcal{V}(\Sigma_1)$ iff there exists a regular u_1 such that

$$T(p)P_0(p)u_1 + C(pI - A)^{-1}x_0 = 0. \tag{4.19}$$

Since the regularity of u implies the regularity of $u_1 := P_0^{-1}(p)u$, the inclusion (i) follows.

(ii): $x_0 \in \mathcal{W}(\Sigma)$ iff there exists an impulsive $u \in \mathcal{U}_\Sigma$ such that $x_0 = x(0+)$, where

$$x = (pI - A)^{-1}Bu, \tag{4.20}$$

and $x_0 \in \mathcal{W}(\Sigma_1)$ iff there exists an impulsive $u_1 \in \mathcal{U}_{\Sigma_1}$ such that $x_0 = x_1(0+)$, where

$$x_1 = (pI - A)^{-1}Bu - \tilde{B}v \tag{4.21}$$

and v and u are as in the definition of Σ_1 (i.e., $u = P_0(p)u_1$, $u_1 = [\bar{u}, v]$). This can be seen either from (4.8) or from $x_1 = (pI - A)^{-1}B_1u_1$. Since u impulsive implies that $u_1 = P_0(p)u$ is impulsive and since $\operatorname{im}\tilde{B} = \mathcal{W}_1 \supseteq \mathcal{W}$, it follows that $x_1(0+) \in \mathcal{W}(\Sigma)$ if $x_1(0+) \in \mathcal{W}(\Sigma_1)$.

(iii): $x_0 \in \mathcal{V}(\Sigma) + \mathcal{W}(\Sigma)$ iff there exists $u \in \mathcal{U}_\Sigma$ such that (4.18) holds (see Proposition 3.13). Similarly, $x_0 \in \mathcal{V}(\Sigma_1) + \mathcal{W}(\Sigma_1)$ iff there exists $u_1 \in \mathcal{U}_{\Sigma_1}$ satisfying (4.19). Since $u_1 \in \mathcal{U}_{\Sigma_1}$ iff $u = P_0(p)u_1 \in \mathcal{U}_\Sigma$ [see (4.15)], the result follows. ■

The formula $D_1 = [\bar{D}, C\tilde{B}]$ and the fact that $\operatorname{rank} D = \operatorname{rank} \bar{D}$ imply that $\operatorname{rank} D_1 \geqslant \operatorname{rank} D$. If $\operatorname{rank} D_1 \neq m$ we can repeat the above procedure. Thus we obtain a sequence of systems $\Sigma_0 := \Sigma, \Sigma_1, \Sigma_2, \dots$ and corresponding matrices D_i such that $\operatorname{rank} D_i \leqslant \operatorname{rank} D_{i+1}$. In addition we have the following inclusions:

$$\mathcal{V}(\Sigma_0) \subseteq \mathcal{V}(\Sigma_1) \subseteq \cdots, \tag{4.22}$$

$$\mathcal{W}(\Sigma_0) \supseteq \mathcal{W}(\Sigma_1) \supseteq \cdots, \tag{4.23}$$

while $\mathcal{V}(\Sigma_i) + \mathcal{W}(\Sigma_i)$ does not depend on i. After one step, the matrix D_1 has the form $D_1 = [\bar{D}, C\tilde{B}]$. Since the columns of $\bar{D}$ are linearly independent, it is possible (and convenient) to choose the matrix S_1 (i.e., the S matrix of the algorithm in the second step) of the form

$$S_1 = \begin{bmatrix} I & S_{11} \\ 0 & S_{12} \end{bmatrix} \tag{4.24}$$

where S_{12} is nonsingular.

After a finite number of steps, the rank of D_i does not increase any more and we have the following result:

THEOREM 4.24. *$\exists \nu \leqslant n$ such that* $\operatorname{rank} D_\nu = \operatorname{rank} D_i$ *for $i \geqslant \nu$, and* $\operatorname{rank} D_\nu = \operatorname{rank} \bar{D}_\nu = \operatorname{rank} T(s)$. *In particular, if Σ is left invertible, then there exists $\nu \leqslant n$ such that* $\operatorname{rank} D_\nu = m$ *(the number of inputs), so that D_ν is left invertible.*

Proof. By duality with the left-structure algorithm and the corresponding result in [14]. ■

Proposition 4.17(ii) can be refined to

$$\mathcal{W}(\Sigma) = \mathcal{W}_1(\Sigma) + \mathcal{W}(\Sigma_1), \tag{4.25}$$

as follows from the same proof [see (4.20) and (4.21)]. Iterating this equality, we obtain

$$\mathcal{W}(\Sigma) = \mathcal{W}_1(\Sigma_0) + \mathcal{W}_1(\Sigma_1) + \cdots + \mathcal{W}_1(\Sigma_\nu) + \mathcal{W}(\Sigma_{\nu+1}) \tag{4.26}$$

The spaces $\mathcal{W}_1(\Sigma_i)$ are computed durirg the structure algorithm [see (4.16)]

$$\mathcal{W}_1(\Sigma_i) = \operatorname{im} \tilde{B}(\Sigma_i) =: \operatorname{im} \tilde{B}_i. \tag{4.27}$$

If D has full column rank, then it is easily seen that $\mathcal{W}(\Sigma) = 0$. In fact, in this case the transfer function $T(s)$ has a proper left inverse $R(s)$, and hence $y = T(p)u$ regular implies that $u = R(s)y$ is regular. Hence $\mathcal{U}_\Sigma$ consists only of regular inputs and $\mathcal{W}(\Sigma) = 0$.

Combining the above results, we have

THEOREM 4.28. *Let Σ be left invertible. Then in the structure algorithm there exists $\nu \leqslant n$ such that* $\operatorname{rank} D_\nu = m$. *The space $\mathcal{W}(\Sigma)$ is given by*

$$\mathcal{W}(\Sigma) = \mathcal{W}_1(\Sigma_0) + \mathcal{W}_1(\Sigma_1) + \cdots + \mathcal{W}_1(\Sigma_{\nu-1}) = \operatorname{im}\left[\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_{\nu-1}\right]$$

Note that for consistency we have attached the index 0 to quantities related to the original system.

5. OPTIMAL OPEN-LOOP CONTROL FOR LEFT-INVERTIBLE SYSTEMS

If Σ is left invertible, then, as noted in the previous section, there exists ν such that D_ν is left invertible. Therefore $\mathcal{U}_{\Sigma_\nu}$ consists only of regular inputs. Iterating the relation (4.15), we obtain

$$\mathcal{U}_\Sigma = \{ P(p)v | v \text{ is regular}\}, \tag{5.1}$$

where

$$P(s) = P_0(s)P_1(s)\cdots P_{\nu-1}(s). \tag{5.2}$$

The relation (5.1) gives an explicit formula for $\mathcal{U}_\Sigma$.

We can use the computation of the previous section to find the optimal control. Using the transformations given in Section 4, we see that the optimization problem as described in Section 2 is equivalent to the following: Determine $u_\nu(t)$ such that for the function $y(t)$ defined by

$$\begin{aligned} px_\nu &= Ax_\nu + B_\nu u_\nu + x_0, \\ y &= Cx_\nu + D_\nu u_\nu, \end{aligned} \tag{5.3}$$

the integral $\int_0^T |y|^2\, dt$ is minimized. Since this is a nonsingular problem, we know that there exists a unique optimal control u_ν^* which is given by

$$u_\nu^*(t) = F_\nu(t)x_0(t), \tag{5.4}$$

where

$$F_\nu(t) := -(D_\nu' D_\nu)^{-1}(B_\nu' K(t) + D_\nu' C)$$

and $K(t)$ is the solution of the Riccati equation

$$\begin{aligned} \dot K(t) &= -C'C - A'K - KA + (KB_\nu + C'D_\nu)(D_\nu' D_\nu)^{-1}(KB_\nu + C'D_\nu)', \\ K(T) &= 0. \end{aligned} \tag{5.5}$$

In addition, the minimal value of J equals

$$J(x_0, u_\nu^*, T) = x_0' K(0) x_0. \tag{5.6}$$

It follows that the optimal control of the original problem exists (within the class $\mathcal{U}_\Sigma$), is unique, and is given by

$$u^* = P(p)u_v^*. \tag{5.7}$$

The solution $K(t)$ of (5.5) is obviously in C^∞ for $t \geqslant 0$, and hence u_v^* is in C^∞ for $t \geqslant 0$, so that u^* is in $\mathcal{C}_{\text{imp}}$. Hence we have

THEOREM 5.8. *The optimal control problem as defined in Section 2 has a unique solution u^*, given by* (5.7), *within the class $\mathcal{U}_\Sigma$. This input is of the form*

$$u^* = u^*_{\text{imp}} + u^*_{\text{reg}}, \tag{5.9}$$

where u^*_{imp} is impulsive and u^*_{reg} is regular.

Note that $u_v^*(t)$ was originally defined only on the interval $[0, T]$. We extend its domain by setting $u_v^*(t) = 0$ for $t < 0$ and by choosing any smooth extension of u_v^* for $t \geqslant T$. This we have to do in order to identify u_v^* with a distribution.

The minimal value of J equals $x_0' K(0) x_0$, as in the modified problem [see (5.6)]. We want to discuss the relevance of the quantity $x_0' K(0) x_0$ for the original problem where we do not use a distributional input.

THEOREM 5.10.

$$x_0' K(0) x_0 = \inf J(x_0, u, T),$$

where the infimum is taken over all C^∞ functions u defined on $[0, T]$.

Proof. Choose a function $\phi \in C^\infty(\mathbb{R})$ satisfying

$$\operatorname{supp} \phi \subseteq [0, \varepsilon], \qquad 0 \leqslant \phi \leqslant \frac{2}{\varepsilon}, \quad \int_0^\varepsilon \phi \, dt = 1.$$

Here $\varepsilon > 0$ is any number. Define

$$v_\phi := \phi * u^* = \phi u^*.$$

(Recall that we denote convolution by juxtaposition.) Then it is well known that v_ϕ is in C^∞. Let y_ϕ denote the output corresponding to the input v_ϕ (and initial state x_0). Then

$$y_\phi = T(p)v_\phi + C(pI - A)^{-1}x_0.$$

Hence,

$$y_\phi - y^* = (\phi - 1)T(p)u^* = (\phi - 1)Z^*$$

because convolution is commutative. Here y^* is the output corresponding to the input u^* and $Z^* := T(p)u^* = y^* - C(pI - A)^{-1}x_0$. It follows that

$$\begin{aligned} |y_\phi - y| &= \left|\int_0^\varepsilon \phi(\tau)Z^*(t-\tau)\,d\tau - Z^*(t)\right| \\ &= \left|\int_0^\varepsilon \{\phi(\tau)Z^*(t-\tau) - Z^*(t)\}\,d\tau\right| \\ &\leqslant \max_{0 \leqslant \tau \leqslant \varepsilon} |Z^*(t-\tau) - Z^*(t)|. \end{aligned}$$

Since Z^* is uniformly continuous on $[0, T]$, it follows that $y_\phi \to y^*$ $(\varepsilon \to 0)$ uniformly with respect to $t \in [0, T]$. In particular,

$$\int_0^T |y_\phi(t)|^2\,dt \to \int_0^T |y^*|^2\,dt \qquad (\varepsilon \to 0).$$ ■

It is well known that $\inf J$ is a quadratic function of x_0 (see [1]). Theorem 5.10 gives a method for computing this infimum explicitly as well as for finding a minimizing control sequence, i.e., a sequence of control functions u_k such that $J(x_0, u_k, T) \to \inf_u J(x_0, u, T)$ for $k \to \infty$.

In Definition 2.1 we introduced the space of states weakly unobservable on $[0, T]$. Let us, for the time being, denote this space by $\mathcal{V}_T$. In Definition 3.8 we introduced the space $\mathcal{V}$ of weakly unobservable states (on $[0, \infty)$). From the definitions it is clear that $\mathcal{V} \subseteq \mathcal{V}_T$ and if $T_1 < T_2$ then $\mathcal{V}_{T_1} \supseteq \mathcal{V}_{T_2}$. Therefore $\mathcal{V}_{0+} := \bigcup_T \mathcal{V}_T$ is a linear subspace satisfying $\mathcal{V}_{0+} \supseteq \mathcal{V}_T \supseteq \mathcal{V}$. We want to show that we have actually $\mathcal{V}_{0+} = \mathcal{V}$ and consequently $\mathcal{V}_T = \mathcal{V}$ for all $T > 0$. Let $x_0 \in \mathcal{V}_{0+}$; then there exists $T > 0$ such that $x_0 \in \mathcal{V}_T$ and an input $u: [0, T] \to \mathbb{R}^m$ such that $y(t) = 0$ for $0 \leqslant t \leqslant T$. As in Proposition 3.9, it follows that $x(t) \in \mathcal{V}_{T-t}$ for $0 < t < T$. Consequently, $x(t) \in \mathcal{V}_{0+}$ for $0 < t < T$. Hence $\dot{x}(0) \in \mathcal{V}_{0+}$. That is, for every $x_0 \in \mathcal{V}_{0+}$ there exists $u_0 \in \mathbb{R}^m$ such that

$Ax_0 + Bu_0 \in \mathcal{V}_{0+}$, $Cx_0 + Du_0 = 0$. In other words, (3.11) is satisfied with $\mathcal{L}$ replaced by $\mathcal{V}_{0+}$. Exactly as in the proof of Theorem 3.10, we infer from this the existence of a feedback F such that $(A + BF)\mathcal{V}_{0+} \subseteq \mathcal{V}_{0+}$, $(C + DF)\mathcal{V}_0 = 0$. Since $\mathcal{V}$ is the largest space for which such a feedback exists, we must have $\mathcal{V}_{0+} \subseteq \mathcal{V}$.

It follows that there exists a regular input such that $J = 0$ iff $x_0 \in \mathcal{V}$. In particular $\mathcal{V} \subseteq \ker K(0)$. However, we do not necessarily have equality, since $K(0)x_0 = 0$ if $\inf J = 0$. The following result follows easily by applying the previous considerations to Σ_ν:

THEOREM 5.11. $\ker K(0) = \mathcal{V} + \mathcal{W}$, *the space of distributionally weakly unobservable states.*

In fact, since $\mathcal{U}(\Sigma_\nu) = 0$, we have by Proposition 4.17

$$\mathcal{V} + \mathcal{W} = \mathcal{V}(\Sigma_\nu) + \mathcal{W}(\Sigma_\nu) = \mathcal{V}(\Sigma_\nu).$$

6. THE INFINITE-HORIZON PROBLEM

Our objective is to minimize $\int_0^\infty |y|^2\,dt$. In order to guarantee the existence of a control for which this integral converges, we assume that Σ is stabilizable. For the standard problem the theory is well established.

THEOREM 6.1. *Suppose that D is left invertible and $C'D = 0$. Then, if (A, B) is stabilizable, the solution of the equation*

$$\dot{P}(t) = C'C + A'P + PA - PB(D'D)^{-1}B'P, \qquad P(0) = 0 \tag{6.2}$$

is nondecreasing and converges to the matrix P_0 that is the smallest nonnegative semidefinite solution of the algebraic Riccati equation

$$C'C + A'P_0 + P_0A - P_0B(D'D)^{-1}B'P_0 = 0. \tag{6.3}$$

The infinite-horizon problem of minimizing $\int_0^\infty |y|^2\,dt$, where y is given by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \qquad x(0) = x_0, \tag{6.4}$$

is solved by the feedback control

$$u = -(D'D)^{-1}B'P_0x. \tag{6.5}$$

Furthermore, the resulting coefficient matrix

$$A_0 := A - (D'D)^{-1}B'P_0 \tag{6.6}$$

is asymptotically stable iff the system is detectable. In this case (6.3) *has a unique nonnegative solution.*

For a proof we refer to [5]. Note that the matrix $K(t)$ of (5.5) is related to $P(t)$ by the formula $K(t,T) = P(T-t)$. Now consider the general nonsingular problem. We have seen that this problem can be reduced to the standard problem by a suitable feedback transformation $u = Fx + v$. Thus we can appeal to Theorem 6.1. The conditions playing a role in this theorem, viz. the stabilizability and detectability, have to be formulated in terms of the new system Σ_F, obtained after the feedback transformation. Of course it is desirable to formulate these conditions in terms of the original system Σ. It is well known that stabilizability properties do not change under feedback transformations. Hence, the stabilizability of Σ_F is equivalent to that of Σ. Detectability, however, is not invariant under feedback. We need a different concept: strong detectability. This concept, first introduced for discrete-time systems in [11], will be defined here for general left-invertible systems (not necessarily with D left invertible).

DEFINITION 6.7. The system Σ:

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

is called *strongly detectable* if for any initial state x_0 and any control u such that $y(t) = 0$ for all $t > 0$ we have $x(t) \to 0$ $(t \to \infty)$.

Plain detectability can be formulated as the property that $u(t) = 0$, $y(t) = 0$ for all $t > 0$ implies $x(t) \to 0$. Obviously strong detectability is a strengthening of detectability.

A number of criteria for strong detectability can be given. Some of these criteria can be found in [11] for discrete-time systems, and in [16] for continuous-time systems.

THEOREM 6.8. *Let Σ be left invertible. The following statements are equivalent*:

(1) *Σ is strongly detectable.*

(2) $\operatorname{rank}\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix} = n + m$ *for* $\operatorname{Re}\lambda \geqslant 0$.

(3) *For all F the pair $(C + DF, A + BF)$ is detectable.*

(4) *If F is such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C + DF)\mathcal{V} = 0$ (compare Theorem* 3.10), *then $A + BF|\mathcal{V}$ is stable.*

(5) *If Σ is detectable and in addition $y(t) = 0$ $(t > 0)$, then $u(t) \to 0$ $(t \to \infty)$.*

Proof. (1) $\Rightarrow$ (2): Obviously, if the strong-detectability condition in Definition 6.7 holds for real functions u, x, y, it also holds for complex functions. Let the rank condition of (2) be violated at λ_0. Then there exist x_0, u_0, not both zero, such that $(\lambda_0 I - A)x_0 - Bu_0 = 0$, $Cx_0 + Du_0 = 0$. It follows that $x_0 \neq 0$, since otherwise $Bu_0 = Du_0 = 0$ would imply $u_0 = 0$ (by the left invertibility of Σ, we must have $\operatorname{rank}[B', D'] = m$). Then if $x(0) = x_0$, $u(t) = e^{\lambda_0 t}u_0$, we have that $x(t) := e^{\lambda_0 t}x_0$ is the state trajectory and $y(t) = 0$ for all t. By the strong-detectability condition we must have $x(t) \to 0$ and hence $\operatorname{Re}\lambda_0 < 0$.

(2) $\Rightarrow$ (3): Since

$$\begin{bmatrix} \lambda I - A - BF & -B \\ C + DF & D \end{bmatrix} = \begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix}$$

has full column rank for every F, the first n columns of this matrix must have full column rank for $\operatorname{Re}\lambda \geqslant 0$. Hence $(A + BF, C + DF)$ is detectable (see [3]).

(3) $\Rightarrow$ (4): If $(A + BF)\mathcal{V} \subseteq \mathcal{V}$ and $(C + DF)\mathcal{V} = 0$, then $\mathcal{V}$ is contained in the $(C + DF, A + BF)$-unobservable space. For $(C + DF, A + BF)$ to be detectable we must have that $A + BF|\mathcal{V}$ is stable.

(4) $\Rightarrow$ (1): Let $y(t) = 0$ for $t > 0$ for some $x_0 \in \mathcal{X}$ and some u. Then, by Definition 3.8, $x_0 \in \mathcal{V}$. The feedback input $\bar{u} := Fx$ yields the state $x(t) = e^{(A+BF)t}x_0$, which converges to zero, since $x_0 \in \mathcal{V}$ and $A + BF|\mathcal{V}$ is stable. The output corresponding to the input $\bar{u}$ equals $(C + DF)x(t) = 0$, since $x(t) \in \mathcal{V}$ for all $t > 0$ (see Proposition 3.9). Since with the same initial state x_0 and with the inputs $\bar{u}$ and u we get zero output, and since the system is left invertible, we must have $\bar{u} = u$.

(5) $\Rightarrow$ (1): For a detectable system $u(t) \to 0$ $(t \to \infty)$, $y(t) = 0$ $(t > 0)$ implies $x(t) \to 0$. [Choose G such that $A + GC$ is stable and observe that $\dot{x} = (A + GC)x + (B + GD)u$.]

(1), (4) $\Rightarrow$ (5): Strong detectability obviously implies detectability. The second statement follows from the proof of (4) $\Rightarrow$ (1). ■

THEOREM 6.9. *Suppose that D is left invertible. Then the optimal feedback stabilizes the system if and only if Σ is strongly detectable.*

Proof. "*Only if*": Suppose that for some x_0 and u we have $y(t) = 0$. Then this function u is obviously the optimal control $u = Fx$, which stabilizes the system. Hence $x(t) \to 0$.

"*If*": Transform the nonsingular problem to the standard problem by means of a feedback transformation $u = F_0 x + v$ (see Remark 2.7). According to Theorem 6.8(3) the resulting system will be detectable. Hence the optimal feedback $v = F_1 x$ stabilizes it. Then $F: = F_0 + F_1$ is the optimal feedback for the original system, and it also yields a stable system. ■

In order to extend this result to singular problems we have to reduce the singular problem to a nonsingular problem, as described in Sections 4 and 5, and to apply Theorem 6.9 to the system thus obtained. However, if we proceed as described in Section 4, we obtain a nonsingular system which is not strongly detectable, even if the original system is. Therefore, we introduce a modification of the structure algorithm of Section 4. The modification is made in order to achieve that the transformations $P_\nu(s)$ (see 4.12) are nonsingular for $\operatorname{Re} s \geqslant 0$. Instead of Equation $(4.n)$ we use Equation $(4.n')$, where

$$\tilde{u} = (p + \alpha)v, \tag{4.6'}$$

with $\alpha > 0$ arbitrary, and

$$p(x - \tilde{B}v) = Ax + \bar{B}\bar{u} + \alpha\tilde{B}v + x_0. \tag{4.7'}$$

In (4.10) we now have

$$B_1: = \left[\bar{B}, (A + \alpha I)\tilde{B}\right].$$

Finally,

$$P_0 = S\begin{bmatrix} I & 0 \\ 0 & (s + a)I \end{bmatrix}. \tag{4.13'}$$

It is easily seen that the statements of Proposition 4.17 and Theorems 4.24, 4.28 remain valid after this modification.

We now have:

THEOREM 6.10. *Σ_1 is stabilizable iff Σ is stabilizable. Σ_1 is strongly detectable iff Σ is strongly detectable.*

Proof. (A, B) is stabilizable iff for $\operatorname{Re}\lambda \geqslant 0$ we have that $\eta A = \lambda\eta$, $\eta B = 0$ implies $\eta = 0$ (see [3]) for every row vector η. Suppose this is the case, and assume $\eta A = \lambda\eta$, $\eta B_1 = 0$ for some λ with $\operatorname{Re}\lambda \geqslant 0$. Then $\eta B_1 = [\eta\bar{B}, (\lambda+\alpha)\eta\tilde{B}] = 0$ and hence $\eta\bar{B} = 0$, $\eta\tilde{B} = 0$, so that $\eta B = 0$. Consequently, $\eta = 0$. The converse is shown similarly.

In order to show the statement about strong detectability we use condition (2) of Theorem 6.8. Suppose that for some λ with $\operatorname{Re}\lambda \geqslant 0$,

$$\operatorname{rank}\begin{bmatrix} \lambda I - A & -B_1 \\ C & D_1 \end{bmatrix} < n + m.$$

Then there exist vectors p, q such that

$$(\lambda I - A)p - B_1 q = 0, \qquad Cp + D_1 q = 0.$$

That is,

$$(\lambda I - A)p - \bar{B}\bar{q} - (A + \alpha I)\tilde{B}\tilde{q} = 0,$$

$$Cp + \bar{D}\bar{q} + C\tilde{B}\tilde{q} = 0,$$

where we have introduced the decomposition $q' =: [\bar{q}', \tilde{q}']$. Substitution of $p + \tilde{B}\tilde{q} = r$ yields

$$\begin{bmatrix} \lambda I - A & -[\bar{B}, & (\lambda+\alpha)\tilde{B}] \\ C & [\bar{D}, & 0] \end{bmatrix}\begin{bmatrix} r \\ \bar{q} \\ \tilde{q} \end{bmatrix} = 0.$$

Hence

$$\begin{bmatrix} \lambda I - A & -B \\ C & D \end{bmatrix}\begin{bmatrix} r \\ q_1 \end{bmatrix} = 0,$$

where

$$q_1 := S\begin{bmatrix} \bar{q} \\ (\lambda+\alpha)^{-1}\tilde{q} \end{bmatrix}$$

Since Σ is strongly detectable, it follows that $r = 0$, $q_1 = 0$. Hence $p = 0$, $q = 0$. ■

Let Σ be a left-invertible, stabilizable, and strongly detectable system. The structure algorithm yields a stabilizable and strongly detectable system Σ_ν with left-invertible D_ν. According to Theorem 6.9, the optimal feedback $u = Fx$, where $F = -(D_\nu' D_\nu)^{-1} B_\nu' P_0$ [see (6.5)], stabilizes Σ_ν, so that the optimal trajectory $x_\nu(t)$ and the optimal control $u_\nu^*(t)$ tend to zero as $t \to \infty$. More specifically, x_ν and u_ν^* satisfy linear differential equations and hence tend to zero exponentially fast [i.e., $|x_\nu(t)| \leqslant Le^{-\gamma t}$, $|u_\nu(t)| \leqslant Le^{-\gamma t}$ for some positive L, γ], and so do their derivatives. It follows that also $u^*(t)$ and $x(t)$, the optimal control and the optimal trajectory, both tend to zero exponentially fast [see e.g. (5.7)]. However, the optimal control is given as an open-loop control, so that one cannot say that Σ has been stabilized by the optimal control. Recall (see Theorem 5.8) that the optimal control consists of an impulsive part and a regular part. What we would like to prove is that the regular part of u^* can be implemented as a state feedback. A direct proof of this, based on the computations of Sections 4 and 5, turns out to be quite complicated. Therefore we prefer to present a somewhat indirect method. As a by-product we will derive a generalization of a result due to Kwakernaak and Sivan [6]. First we give an infinite-horizon version of Theorem 5.10. From now on, it will be our standing assumption that Σ is left invertible, stabilizable, and strongly detectable.

THEOREM 6.12. *Assume that Σ left invertible, stabilizable, and strongly detectable. Let P be the (unique) nonnegative solution of the Riccati equation of system Σ_ν. Then*

$$x_0' P x_0 = \inf \int_0^\infty |y|^2 \, dt,$$

where the infimum is taken over all smooth u satisfying $\int_0^\infty |u|^2 \, dt < \infty$.

Proof. Let $u^* \in \mathcal{U}_\Sigma$ minimize $\int_0^\infty |y|^2 \, dt$, and let x^* and y^* denote the corresponding trajectory and output. Choose a matrix F such that $A + BF$ is stable. Then

$$u^* = Fx^* + w^*,$$

where $w^* := u^* - Fx^*$. It follows that

$$y^* = T_F(p)w^* + C_F(pI - A_F)^{-1}x_0,$$

where $A_F := A + BF$, $C_F := C + DF$, $T_F(p) := C_F(pI - A_F)^{-1}B + D$.

We notice that $|\dot{w}^*(t)| \leqslant le^{-\gamma t}$ for $t > 0$, for some $l > 0$, $\gamma > 0$. For an approximate optimal control we choose

$$v := Fx + \phi w^*.$$

It is easily seen that v is regular, since ϕw^* is regular. Denoting the corresponding output by y_ϕ, we obtain $y_\phi - y^* = \phi Z$, where

$$Z = T_F(p)w^*$$

Hence

$$|y_\phi - y^*| \leqslant \int_0^\varepsilon \phi(\tau)|Z(t-\tau) - Z(t)|\,dt$$

$$\leqslant \max_{0 \leqslant \tau \leqslant \varepsilon} |Z(t-\tau) - Z(t)| \leqslant \varepsilon l_0 e^{-\gamma_0 t}$$

where we have used the fact that, because of the stability of the transfer matrix $T_F(s)$ and the fact that $|\dot{w}^*(t)| \leqslant le^{-\gamma t}$, we have $|\dot{Z}(t)| \leqslant l_0 e^{-\gamma_0 t}$. It follows that

$$\int_0^\infty |y_\phi(t)|^2\,dt \leqslant \int_0^\infty |y|^2\,dt + l_1\varepsilon^2.$$

In addition, $\int_0^\infty |\nu|^2\,dt < \infty$, since $|v^*(t)| \leqslant Le^{-\gamma t}$ for some $L > 0$. ■

Next we give a generalization of the Kwakernaak-Sivan result.

COROLLARY 6.13. *Assume that* Σ *is left invertible, stabilizable, and strongly detectable. Let* P *be as in Theorem* 6.12, *and define* P_ε *for* $\varepsilon > 0$ *by*

$$x_0' P_\varepsilon x_0 = \min_u \int_0^\infty \left(|y|^2 + \varepsilon^2|u|^2\right)dt$$

Then

$$P_\varepsilon \downarrow P \qquad (\varepsilon \downarrow 0).$$

Proof. Choose u such that $\|u\|^2 := \int_0^\infty |u|^2\,dt < \infty$ and $\|y\|^2 := \int_0^\infty |y|^2\,dx < x_0' P x_0 + \varepsilon_1$, where $\varepsilon_1 > 0$ is given (see Theorem 6.12). Then

$$\begin{aligned} x_0' P_\varepsilon x_0 &\leqslant \|y\|^2 + \varepsilon\|u\|^2 \leqslant x_0' P x_0 + \varepsilon\|u\|^2 + \varepsilon_1 \\ &\leqslant x_0' P x_0 + 2\varepsilon_1 \end{aligned}$$

if ε is sufficiently small. Since ε_1 is arbitrary, we have $x_0' P_\varepsilon x_0 \to x_0' P x_0$ for $\varepsilon \downarrow 0$. ■

It follows that the performance of the impulsive-smooth open-loop optimal control can be approximated arbitrarily closely by the optimal feedback F_ε of the problem of Corollary 6.13. For truly singular problems, F_ε must be "high gain" (compare [12, VIII D]).

REMARK 6.14. In [6] much attention is spent on the special case $P = 0$. In Theorem 5.11 it is shown that $\ker K(0) = \mathcal{V} + \mathcal{W}$. It follows that also $\ker P = \mathcal{V} + \mathcal{W}$. Hence $P = 0$ iff $\mathcal{V} + \mathcal{W} = \mathcal{X}$, or equivalently iff Σ is right invertible (see Theorem 3.24). General conditions for $\lim J_\varepsilon(x_0)$ to be zero (under the assumption $D = 0$) were given in [14]. Theorem 5.11 is a generalization in a different direction.

For any regular u such that $x(t) \to 0$ we have

$$\int_0^\infty |y|^2\,dt - x_0' P x_0 = \int_0^\infty \left(|y|^2 + \frac{d}{dt} xPx \right) dt = \int_0^\infty [\,x' \quad u'\,] N \begin{bmatrix} x \\ u \end{bmatrix} dt,$$

where

$$N := \begin{bmatrix} C'C + A'P + PA & C'D + PB \\ D'C + B'P & D'D \end{bmatrix}. \tag{6.15}$$

LEMMA 6.16. $N \geqslant 0$.

Proof. If D is left invertible, P satisfies the algebraic Riccati equation

$$C'C + A'P + PA - (C'D + PB)(D'D)^{-1}(C'D + PB)' = 0 \tag{6.17}$$

If we introduce

$$E: = C'C + A'P + PA,$$
$$H: = C'D + PB, \qquad L: = D'D$$

then (6.17) reads $E = HL^{-1}H'$. Since $L > 0$, it follows that

$$N = \begin{bmatrix} E & H \\ H' & L \end{bmatrix} \geqslant 0$$

—specifically,

$$[x', u']N\begin{bmatrix} x \\ u \end{bmatrix} = (H'x + Lu)'L^{-1}(H'x + Lu).$$

In the general case (when D is not left invertible), we define for $\varepsilon > 0$

$$D_\varepsilon: = \begin{bmatrix} D \\ \varepsilon I \end{bmatrix}, \qquad C_\varepsilon: = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

Then

$$N_\varepsilon = \begin{bmatrix} C'C + A'P_\varepsilon + P_\varepsilon A & C'D + P_\varepsilon B \\ D'C + B'P_\varepsilon & D'D + \varepsilon^2 I \end{bmatrix} \geqslant 0,$$

where P_ε is characterized by

$$x_0' P_\varepsilon x_0 = \min_u \int_0^\infty \left(|y|^2 + \varepsilon|y|^2\right) dt.$$

Since $P_\varepsilon \to P$ for $\varepsilon \downarrow 0$ (see Corollary 6.13), we have $N_\varepsilon \to N$ and hence $N \geqslant 0$. ∎

Because of Lemma 6.16 we can factorize N, say

(6.18) $$N = \begin{bmatrix} \hat{C}' \\ \hat{D}' \end{bmatrix} [\hat{C} \quad \hat{D}],$$

which gives rise to a new system $\hat{\Sigma}$:

(6.19) $$\dot{x} = Ax + Bu, \qquad \hat{y} = \hat{C}x + \hat{D}u.$$

For regular inputs such that $x(t) \to 0$ $(t \to \infty)$ it follows from the foregoing computation that

(6.20) $$\int_0^\infty |\hat{y}|^2 \, dt = \int_0^\infty |y|^2 \, dt - x_0' P x_0.$$

LEMMA 6.21. *$\mathcal{U}_{\hat{\Sigma}} = \mathcal{U}_{\Sigma}$, and the relation* (6.20) *is valid for all $u \in \mathcal{U}_{\Sigma}$ such that $x(t) \to 0$ $(t \to \infty)$.*

Proof. Choose a sequence of smooth functions ϕ_i with support on $[0, \infty)$ such that ϕ_i converges weakly in $\mathfrak{D}'_+$ to δ, i.e., $(\phi_i, \phi) \to \phi(0)$ $(i \to \infty)$. Let $u \in \mathcal{U}_{\Sigma}$. Let $t_1 > 0$ be any number.

We choose $u_0 \in \mathcal{U}_{\Sigma}$ such that $u = u_0$ on $(-\infty, t_1]$ and $|u_0(t)| \leqslant Le^{-\gamma t}$, $x(t) \leqslant Le^{-\gamma t}$ for some $\gamma > 0$, $L > 0$. (We assume zero initial state.) This can be achieved by choosing any u_1 such that the corresponding state trajectory x_1 as well as u_1 satisfies the desired inequality, and defining $u_0 = \psi \circ u + \chi \circ u_1$, where ψ, χ are smooth functions with support in $(-\infty, t_2]$ (where $t_2 > t_1$) and in (t_2, ∞), respectively. Here $\circ$ denotes pointwise multiplication (recall that juxtaposition always denotes convolution). Define $u_i := \phi_i u_0$. Then $y_i := T(p)u_i = \phi_i y_0 \to y_0 := T(p)u_0$ in the $\mathcal{L}_2$ sense. In particular

$$\|y_i\|^2 := \int_0^\infty |y_i|^2\,dt \to \int_0^\infty |y|^2\,dt =: \|y\|^2$$

On the other hand, $\hat{y}_i := \hat{T}(p)u_i = \phi_i \hat{y}_0$, where $\hat{y}_0 := \hat{T}(p)u_0$. Here $\hat{T}(s)$ is the transfer function of $\hat{\Sigma}$. It follows that $\hat{y}_i \to \hat{y}_0$ weakly in $\mathfrak{D}'^r_+$. Also, we note that $\|\hat{y}_i\|^2 = \|y_i\|^2$ [see (6.20)], and hence that $\|\hat{y}_i\|^2$ is bounded. It follows that there is a weakly convergent subsequence $\hat{y}_{i_k} \to \tilde{y}$ in $\mathcal{L}_2$. That is, $\langle \hat{y}_{i_k}, \phi \rangle \to \langle \tilde{y}, \phi \rangle$ for every smooth ϕ in $\mathcal{L}_2$—in particular, for every $\phi \in \mathfrak{D}$. Hence $\hat{y}_{i_k} \to \tilde{y}$ weakly, and consequently, $\tilde{y} = \hat{y}_0$. Since $\tilde{y} \in \mathcal{L}_2$, $\hat{y}_0$ cannot have an impulsive part. Finally, $\hat{y}_0 = \hat{y} := \hat{T}(p)u$ on $[0, t_1]$, and consequently, $\hat{y}$ has no impulsive part. Thus, $\mathcal{U}_{\Sigma} \subseteq \mathcal{U}_{\hat{\Sigma}}$. The converse is proved similarly. To prove the second statement, we drop the assumption $x_0 = 0$, and note that $\|\hat{y}\|^2 = \lim \|\hat{y}_i\|^2 = \lim \|y_i\|^2 - x_0' P x_0 = \|y\|^2 - x_0' P x_0$. ∎

As a consequence of Lemma 6.21 we have $\hat{\mathcal{W}} = \mathcal{W}$ and $\hat{\mathcal{V}} \supseteq \mathcal{V}$. The equality holds because $\mathcal{W}$ is expressed completely in terms of A, B, and $\mathcal{U}_{\Sigma}$, and the inclusion follows from (6.20), since $x_0 \in \mathcal{V}$ iff there exists a regular u such that $y(t) = 0$ for all t. Finally, $\inf \|\hat{y}\|^2 = \inf \|y\|^2 - x_0' P x_0 = 0$, and hence according the infinite-horizon version of Theorem 5.11 (see also Remark 6.14), $\hat{\mathcal{V}} + \hat{\mathcal{W}} = \mathcal{X}$. On the other hand, $\hat{\Sigma}$ is left invertible. This follows from $\mathcal{U}_{\hat{\Sigma}} = \mathcal{U}_{\Sigma}$. If $\hat{\Sigma}$ were not left invertible, there would exist a polynomial vector $\psi(s)$ such that $T(s)\psi(s) = 0$. Then $\psi(p)p^k$ would be in $\mathcal{U}_{\hat{\Sigma}}$ for all k. This would contradict (5.1). Since $\hat{\Sigma}$ is left invertible, we have $\hat{\mathcal{V}} \cap \hat{\mathcal{W}} = 0$, so that

$$\hat{\mathcal{V}} \oplus \hat{\mathcal{W}} = \mathcal{X}. \tag{6.22}$$

Because of (6.20), the optimal controls of Σ and $\hat{\Sigma}$ are the same. For $\hat{\Sigma}$, however, the optimal control can be seen to be as follows: Write $x_0 \in \mathcal{X}$ as $x_0 = v + w$ with $v \in \hat{\mathcal{V}}$, $w \in \hat{\mathcal{W}}$. Choose an impulsive input $u_1 \in \mathcal{U}_\Sigma$ such that $x(0+) = v$, choose F such that $(A + BF)\hat{\mathcal{V}} \subseteq \hat{\mathcal{V}}$, $(C + DF)\hat{\mathcal{V}} = 0$ (see Theorem 3.10), and choose the feedback control $u_2 = Fx$ for $t > 0$. Then $\|\hat{y}\| = 0$, and hence $u = u_1 + u_2$ is optimal for $\hat{\Sigma}$ and therefore also optimal for Σ.

COMMENT 6.23. As in the discrete-time case [11], the condition that Σ is left invertible, stabilizable, and detectable can be interpreted as a minimum-phase condition (stable transmission zeros). Hence, the satisfying result emerges that minimum-phase systems correspond to solvable singular control problems with stable closed-loop implementation.

REFERENCES

1 D. J. Clements and B. D. O. Anderson, *Singular Optimal Control: The Linear Quadratic Problem*, Springer Lecture Notes in Control and Information Sciences, No. 5, Springer, Berlin, 1978.

2 J. Grasman, Nonuniqueness in singular optimal control, Math. Centre, Amsterdam, 1979.

3 M. L. J. Hautus, Stabilization, controllability and observability of linear autonomous systems, *Nederl. Akad. Wetensch. Proc. Ser. A* 73:448–455 (1970).

4 M. L. J. Hautus, The formal Laplace transform for smooth linear systems, in *Proceedings of the International Symposium on Mathematical Systems Theory*, Udine, 1975, Lecture Notes in Econ. and Math. Systems, No. 131, 1976, pp. 29–46.

5 V. Kucera, A contribution to matrix quadratic equations, *IEEE Trans. Automat. Control* AC-17:344–347 (1972).

6 H. Kwakernaak and R. Sivan, The maximally achievable accuracy of linear optimal regulators and linear optimal filters, *IEEE Trans. Automat. Control* AC-17:79–86 (1972).

7 B. P. Molinari, A strong controllability and observability in linear multivariable control, *IEEE Trans. Automat. Control* AC-21:761–764 (1976).

8 A. S. Morse, Structural invariants of linear multivariable systems, *SIAM J. Control* 9:143–148 (1971).

9 P. J. Moylan and J. B. Moore, Generalizations of singular optimal control theory, *Automatica* 7:591–598 (1971).

10 L. Schwartz, *Theorie des Distributions*, Hermann, Paris, 1978.

11 L. Silverman, Discrete Riccati equations: alternative algorithms, asymptotic properties, and system theory interpretations, in *Control and Dynamic Systems*, Vol. 12, 1970, pp. 381–385.

12 J. C. Willems, Almost invariant subspaces: An approach to high gain feedback design—Part I: Almost controlled invariant subspaces, *IEEE Trans. Automat. Control* AC-26:235–252 (1981).

13 W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, Springer, New York, 1979.

14 B. A. Francis, The optimal linear-quadratic time-invariant regulator with cheap control, *IEEE Trans. Automat. Control* AC-24:616–621 (1979).

15 R. E. O'Malley, Jr. and A. Jaemson, Singular perturbation and singular arcs I, II, *IEEE Trans. Aut. Contr.*, AC-20:218–226 (1975); AC-22:328–337 (1977).

16 M. L. J. Hautus, Strong detectability and strong observers, *Linear Algebra Appl.* 50:353–368 (1983).

Received 5 February 1982; revised 10 August 1982